



Pearson

HASS • HEIL • WEIR

CALCULUS

THOMAS'

FOURTEENTH EDITION

THOMAS' CALCULUS

FOURTEENTH EDITION

Based on the original work by

GEORGE B. THOMAS, JR.

Massachusetts Institute of Technology

as revised by

JOEL HASS

University of California, Davis

CHRISTOPHER HEIL

Georgia Institute of Technology

MAURICE D. WEIR

Naval Postgraduate School



Pearson

Director, Portfolio Management: Deirdre Lynch
Executive Editor: Jeff Weidenaar
Editorial Assistant: Jennifer Snyder
Content Producer: Rachel S. Reeve
Managing Producer: Scott Disanno
Producer: Stephanie Green
TestGen Content Manager: Mary Durnwald
Manager: Content Development, Math: Kristina Evans
Product Marketing Manager: Emily Ockay
Field Marketing Manager: Evan St. Cyr
Marketing Assistants: Jennifer Myers, Erin Rush
Senior Author Support/Technology Specialist: Joe Vetere
Rights and Permissions Project Manager: Gina M. Cheselka
Manufacturing Buyer: Carol Melville, LSC Communications, Inc.
Program Design Lead: Barbara T. Atkinson
Associate Director of Design: Blair Brown
Text and Cover Design, Production Coordination, Composition: Cenveo® Publisher Services
Illustrations: Network Graphics, Cenveo® Publisher Services

Cover Image: Te Rewa Rewa Bridge, Getty Images/Kanwal Sandhu

Copyright © 2018, 2014, 2010 by Pearson Education, Inc. All Rights Reserved. Printed in the United States of America. This publication is protected by copyright, and permission should be obtained from the publisher prior to any prohibited reproduction, storage in a retrieval system, or transmission in any form or by any means, electronic, mechanical, photocopying, recording, or otherwise. For information regarding permissions, request forms and the appropriate contacts within the Pearson Education Global Rights & Permissions department, please visit www.pearsoned.com/permissions/.

Attributions of third party content appear on page C-1, which constitutes an extension of this copyright page.

PEARSON, ALWAYS LEARNING, and MYMATHLAB are exclusive trademarks owned by Pearson Education, Inc. or its affiliates in the U.S. and/or other countries.

Unless otherwise indicated herein, any third-party trademarks that may appear in this work are the property of their respective owners and any references to third-party trademarks, logos or other trade dress are for demonstrative or descriptive purposes only. Such references are not intended to imply any sponsorship, endorsement, authorization, or promotion of Pearson's products by the owners of such marks, or any relationship between the owner and Pearson Education, Inc. or its affiliates, authors, licensees or distributors.

Library of Congress Cataloging-in-Publication Data

Names: Hass, Joel. | Heil, Christopher, 1960- | Weir, Maurice D.

Title: Thomas' calculus / based on the original work by George B. Thomas, Jr., Massachusetts Institute of Technology, as revised by Joel Hass, University of California, Davis, Christopher Heil, Georgia Institute of Technology, Maurice D. Weir, Naval Postgraduate School.

Description: Fourteenth edition. | Boston : Pearson, [2018] | Includes index.

Identifiers: LCCN 2016055262 | ISBN 9780134438986 | ISBN 0134438981

Subjects: LCSH: Calculus--Textbooks. | Geometry, Analytic--Textbooks.

Classification: LCC QA303.2.W45 2018 | DDC 515--dc23 LC record available at <https://lcn.loc.gov/2016055262>



Instructor's Edition
ISBN 13: 978-0-13-443909-9
ISBN 10: 0-13-443909-0

Student Edition
ISBN 13: 978-0-13-443898-6
ISBN 10: 0-13-443898-1

Contents

Preface ix

1

Functions 1

- 1.1 Functions and Their Graphs 1
- 1.2 Combining Functions; Shifting and Scaling Graphs 14
- 1.3 Trigonometric Functions 21
- 1.4 Graphing with Software 29
 - Questions to Guide Your Review* 33
 - Practice Exercises* 34
 - Additional and Advanced Exercises* 35
 - Technology Application Projects* 37

2

Limits and Continuity 38

- 2.1 Rates of Change and Tangent Lines to Curves 38
- 2.2 Limit of a Function and Limit Laws 45
- 2.3 The Precise Definition of a Limit 56
- 2.4 One-Sided Limits 65
- 2.5 Continuity 72
- 2.6 Limits Involving Infinity; Asymptotes of Graphs 83
 - Questions to Guide Your Review* 96
 - Practice Exercises* 97
 - Additional and Advanced Exercises* 98
 - Technology Application Projects* 101

3

Derivatives 102

- 3.1 Tangent Lines and the Derivative at a Point 102
- 3.2 The Derivative as a Function 106
- 3.3 Differentiation Rules 115
- 3.4 The Derivative as a Rate of Change 124
- 3.5 Derivatives of Trigonometric Functions 134
- 3.6 The Chain Rule 140
- 3.7 Implicit Differentiation 148
- 3.8 Related Rates 153
- 3.9 Linearization and Differentials 162
 - Questions to Guide Your Review* 174
 - Practice Exercises* 174
 - Additional and Advanced Exercises* 179
 - Technology Application Projects* 182

4

Applications of Derivatives 183

- 4.1 Extreme Values of Functions on Closed Intervals 183
- 4.2 The Mean Value Theorem 191
- 4.3 Monotonic Functions and the First Derivative Test 197
- 4.4 Concavity and Curve Sketching 202
- 4.5 Applied Optimization 214
- 4.6 Newton's Method 226
- 4.7 Antiderivatives 231
- Questions to Guide Your Review* 241
- Practice Exercises* 241
- Additional and Advanced Exercises* 244
- Technology Application Projects* 247

5

Integrals 248

- 5.1 Area and Estimating with Finite Sums 248
- 5.2 Sigma Notation and Limits of Finite Sums 258
- 5.3 The Definite Integral 265
- 5.4 The Fundamental Theorem of Calculus 278
- 5.5 Indefinite Integrals and the Substitution Method 289
- 5.6 Definite Integral Substitutions and the Area Between Curves 296
- Questions to Guide Your Review* 306
- Practice Exercises* 307
- Additional and Advanced Exercises* 310
- Technology Application Projects* 313

6

Applications of Definite Integrals 314

- 6.1 Volumes Using Cross-Sections 314
- 6.2 Volumes Using Cylindrical Shells 325
- 6.3 Arc Length 333
- 6.4 Areas of Surfaces of Revolution 338
- 6.5 Work and Fluid Forces 344
- 6.6 Moments and Centers of Mass 353
- Questions to Guide Your Review* 365
- Practice Exercises* 366
- Additional and Advanced Exercises* 368
- Technology Application Projects* 369

7

Transcendental Functions 370

- 7.1 Inverse Functions and Their Derivatives 370
- 7.2 Natural Logarithms 378
- 7.3 Exponential Functions 386
- 7.4 Exponential Change and Separable Differential Equations 397
- 7.5 Indeterminate Forms and L'Hôpital's Rule 407
- 7.6 Inverse Trigonometric Functions 416
- 7.7 Hyperbolic Functions 428

- 7.8 Relative Rates of Growth 436
Questions to Guide Your Review 441
Practice Exercises 442
Additional and Advanced Exercises 445

8

Techniques of Integration 447

- 8.1 Using Basic Integration Formulas 447
8.2 Integration by Parts 452
8.3 Trigonometric Integrals 460
8.4 Trigonometric Substitutions 466
8.5 Integration of Rational Functions by Partial Fractions 471
8.6 Integral Tables and Computer Algebra Systems 479
8.7 Numerical Integration 485
8.8 Improper Integrals 494
8.9 Probability 505
Questions to Guide Your Review 518
Practice Exercises 519
Additional and Advanced Exercises 522
Technology Application Projects 525

9

First-Order Differential Equations 526

- 9.1 Solutions, Slope Fields, and Euler's Method 526
9.2 First-Order Linear Equations 534
9.3 Applications 540
9.4 Graphical Solutions of Autonomous Equations 546
9.5 Systems of Equations and Phase Planes 553
Questions to Guide Your Review 559
Practice Exercises 559
Additional and Advanced Exercises 561
Technology Application Projects 562

10

Infinite Sequences and Series 563

- 10.1 Sequences 563
10.2 Infinite Series 576
10.3 The Integral Test 586
10.4 Comparison Tests 592
10.5 Absolute Convergence; The Ratio and Root Tests 597
10.6 Alternating Series and Conditional Convergence 604
10.7 Power Series 611
10.8 Taylor and Maclaurin Series 622
10.9 Convergence of Taylor Series 627
10.10 Applications of Taylor Series 634
Questions to Guide Your Review 643
Practice Exercises 644
Additional and Advanced Exercises 646
Technology Application Projects 648

11 Parametric Equations and Polar Coordinates 649

- 11.1 Parametrizations of Plane Curves 649
- 11.2 Calculus with Parametric Curves 658
- 11.3 Polar Coordinates 667
- 11.4 Graphing Polar Coordinate Equations 671
- 11.5 Areas and Lengths in Polar Coordinates 675
- 11.6 Conic Sections 680
- 11.7 Conics in Polar Coordinates 688
 - Questions to Guide Your Review* 694
 - Practice Exercises* 695
 - Additional and Advanced Exercises* 697
 - Technology Application Projects* 699

12 Vectors and the Geometry of Space 700

- 12.1 Three-Dimensional Coordinate Systems 700
- 12.2 Vectors 705
- 12.3 The Dot Product 714
- 12.4 The Cross Product 722
- 12.5 Lines and Planes in Space 728
- 12.6 Cylinders and Quadric Surfaces 737
 - Questions to Guide Your Review* 743
 - Practice Exercises* 743
 - Additional and Advanced Exercises* 745
 - Technology Application Projects* 748

13 Vector-Valued Functions and Motion in Space 749

- 13.1 Curves in Space and Their Tangents 749
- 13.2 Integrals of Vector Functions; Projectile Motion 758
- 13.3 Arc Length in Space 767
- 13.4 Curvature and Normal Vectors of a Curve 771
- 13.5 Tangential and Normal Components of Acceleration 777
- 13.6 Velocity and Acceleration in Polar Coordinates 783
 - Questions to Guide Your Review* 787
 - Practice Exercises* 788
 - Additional and Advanced Exercises* 790
 - Technology Application Projects* 791

- ## 14 Partial Derivatives 792
- 14.1 Functions of Several Variables 792
 - 14.2 Limits and Continuity in Higher Dimensions 800
 - 14.3 Partial Derivatives 809
 - 14.4 The Chain Rule 821
 - 14.5 Directional Derivatives and Gradient Vectors 831
 - 14.6 Tangent Planes and Differentials 839
 - 14.7 Extreme Values and Saddle Points 849
 - 14.8 Lagrange Multipliers 858
 - 14.9 Taylor's Formula for Two Variables 868
 - 14.10 Partial Derivatives with Constrained Variables 872
 - Questions to Guide Your Review* 876
 - Practice Exercises* 877
 - Additional and Advanced Exercises* 880
 - Technology Application Projects* 882

- ## 15 Multiple Integrals 883
- 15.1 Double and Iterated Integrals over Rectangles 883
 - 15.2 Double Integrals over General Regions 888
 - 15.3 Area by Double Integration 897
 - 15.4 Double Integrals in Polar Form 900
 - 15.5 Triple Integrals in Rectangular Coordinates 907
 - 15.6 Applications 917
 - 15.7 Triple Integrals in Cylindrical and Spherical Coordinates 927
 - 15.8 Substitutions in Multiple Integrals 939
 - Questions to Guide Your Review* 949
 - Practice Exercises* 949
 - Additional and Advanced Exercises* 952
 - Technology Application Projects* 954

- ## 16 Integrals and Vector Fields 955
- 16.1 Line Integrals of Scalar Functions 955
 - 16.2 Vector Fields and Line Integrals: Work, Circulation, and Flux 962
 - 16.3 Path Independence, Conservative Fields, and Potential Functions 975
 - 16.4 Green's Theorem in the Plane 986
 - 16.5 Surfaces and Area 998
 - 16.6 Surface Integrals 1008
 - 16.7 Stokes' Theorem 1018
 - 16.8 The Divergence Theorem and a Unified Theory 1031
 - Questions to Guide Your Review* 1044
 - Practice Exercises* 1044
 - Additional and Advanced Exercises* 1047
 - Technology Application Projects* 1048

17**Second-Order Differential Equations** (Online at [www.goo.gl/MgDXPY](http://www.google.com/search?q=www.goo.gl/MgDXPY))

- 17.1** Second-Order Linear Equations
- 17.2** Nonhomogeneous Linear Equations
- 17.3** Applications
- 17.4** Euler Equations
- 17.5** Power-Series Solutions

Appendices AP-1

- A.1** Real Numbers and the Real Line AP-1
- A.2** Mathematical Induction AP-6
- A.3** Lines, Circles, and Parabolas AP-9
- A.4** Proofs of Limit Theorems AP-19
- A.5** Commonly Occurring Limits AP-22
- A.6** Theory of the Real Numbers AP-23
- A.7** Complex Numbers AP-26
- A.8** The Distributive Law for Vector Cross Products AP-34
- A.9** The Mixed Derivative Theorem and the Increment Theorem AP-35

Answers to Odd-Numbered Exercises A-1**Applications Index** AI-1**Subject Index** I-1**A Brief Table of Integrals** T-1**Credits** C-1

Preface



Thomas' Calculus, Fourteenth Edition, provides a modern introduction to calculus that focuses on developing conceptual understanding of the underlying mathematical ideas. This text supports a calculus sequence typically taken by students in STEM fields over several semesters. Intuitive and precise explanations, thoughtfully chosen examples, superior figures, and time-tested exercise sets are the foundation of this text. We continue to improve this text in keeping with shifts in both the preparation and the goals of today's students, and in the applications of calculus to a changing world.

Many of today's students have been exposed to calculus in high school. For some, this translates into a successful experience with calculus in college. For others, however, the result is an overconfidence in their computational abilities coupled with underlying gaps in algebra and trigonometry mastery, as well as poor conceptual understanding. In this text, we seek to meet the needs of the increasingly varied population in the calculus sequence. We have taken care to provide enough review material (in the text and appendices), detailed solutions, and a variety of examples and exercises, to support a complete understanding of calculus for students at varying levels. Additionally, the MyMathLab course that accompanies the text provides adaptive support to meet the needs of all students. Within the text, we present the material in a way that supports the development of mathematical maturity, going beyond memorizing formulas and routine procedures, and we show students how to generalize key concepts once they are introduced. References are made throughout, tying new concepts to related ones that were studied earlier. After studying calculus from *Thomas*, students will have developed problem-solving and reasoning abilities that will serve them well in many important aspects of their lives. Mastering this beautiful and creative subject, with its many practical applications across so many fields, is its own reward. But the real gifts of studying calculus are acquiring the ability to think logically and precisely; understanding what is defined, what is assumed, and what is deduced; and learning how to generalize conceptually. We intend this book to encourage and support those goals.

New to This Edition

We welcome to this edition a new coauthor, Christopher Heil from the Georgia Institute of Technology. He has been involved in teaching calculus, linear algebra, analysis, and abstract algebra at Georgia Tech since 1993. He is an experienced author and served as a consultant on the previous edition of this text. His research is in harmonic analysis, including time-frequency analysis, wavelets, and operator theory.

This is a substantial revision. Every word, symbol, and figure was revisited to ensure clarity, consistency, and conciseness. Additionally, we made the following text-wide updates:

- Updated graphics to bring out clear visualization and mathematical correctness.
- Added examples (in response to user feedback) to overcome conceptual obstacles. See Example 3 in Section 9.1.
- Added new types of homework exercises throughout, including many with a geometric nature. The new exercises are not just more of the same, but rather give different perspectives on and approaches to each topic. We also analyzed aggregated student usage and performance data from MyMathLab for the previous edition of this text. The results of this analysis helped improve the quality and quantity of the exercises.
- Added short URLs to historical links that allow students to navigate directly to online information.
- Added new marginal notes throughout to guide the reader through the process of problem solution and to emphasize that each step in a mathematical argument is rigorously justified.

New to MyMathLab[®]

Many improvements have been made to the overall functionality of MyMathLab (MML) since the previous edition. Beyond that, we have also increased and improved the content specific to this text.

- Instructors now have more exercises than ever to choose from in assigning homework. There are approximately 8080 assignable exercises in MML.
- The MML exercise-scoring engine has been updated to allow for more robust coverage of certain topics, including differential equations.
- A full suite of Interactive Figures have been added to support teaching and learning. The figures are designed to be used in lecture, as well as by students independently. The figures are editable using the freely available GeoGebra software. The figures were created by Marc Renault (Shippensburg University), Kevin Hopkins (Southwest Baptist University), Steve Phelps (University of Cincinnati), and Tim Brzezinski (Berlin High School, CT).
- Enhanced Sample Assignments include just-in-time prerequisite review, help keep skills fresh with distributed practice of key concepts (based on research by Jeff Hieb of University of Louisville), and provide opportunities to work exercises without learning aids (to help students develop confidence in their ability to solve problems independently).
- Additional Conceptual Questions augment text exercises to focus on deeper, theoretical understanding of the key concepts in calculus. These questions were written by faculty at Cornell University under an NSF grant. They are also assignable through Learning Catalytics.
- An Integrated Review version of the MML course contains pre-made quizzes to assess the prerequisite skills needed for each chapter, plus personalized remediation for any gaps in skills that are identified.
- Setup & Solve exercises now appear in many sections. These exercises require students to show how they set up a problem as well as the solution, better mirroring what is required of students on tests.
- Over 200 new instructional videos by Greg Wisloski and Dan Radelet (both of Indiana University of PA) augment the already robust collection within the course. These videos support the overall approach of the text—specifically, they go beyond routine procedures to show students how to generalize and connect key concepts.

Content Enhancements

Chapter 1

- Shortened 1.4 to focus on issues arising in use of mathematical software and potential pitfalls. Removed peripheral material on regression, along with associated exercises.
- Added new Exercises: **1.1:** 59–62, **1.2:** 21–22; **1.3:** 64–65, **PE:** 29–32.

Chapter 2

- Added definition of average speed in 2.1.
- Clarified definition of limits to allow for arbitrary domains. The definition of limits is now consistent with the definition in multivariable domains later in the text and with more general mathematical usage.
- Reworded limit and continuity definitions to remove implication symbols and improve comprehension.
- Added new Example 7 in 2.4 to illustrate limits of ratios of trig functions.
- Rewrote 2.5 Example 11 to solve the equation by finding a zero, consistent with previous discussion.
- Added new Exercises: **2.1:** 15–18; **2.2:** 3h–k, 4f–i; **2.4:** 19–20, 45–46; **2.6:** 69–72; **PE:** 49–50; **AAE:** 33.

Chapter 3

- Clarified relation of slope and rate of change.
- Added new Figure 3.9 using the square root function to illustrate vertical tangent lines.
- Added figure of $x \sin(1/x)$ in 3.2 to illustrate how oscillation can lead to nonexistence of a derivative of a continuous function.
- Revised product rule to make order of factors consistent throughout text, including later dot product and cross product formulas.
- Added new Exercises: **3.2:** 36, 43–44; **3.3:** 51–52; **3.5:** 43–44, 61bc; **3.6:** 65–66, 97–99; **3.7:** 25–26; **3.8:** 47; **AAE:** 24–25.

Chapter 4

- Added summary to 4.1.
- Added new Example 3 with new Figure 4.27 to give basic and advanced examples of concavity.
- Added new Exercises: **4.1:** 61–62; **4.3:** 61–62; **4.4:** 49–50, 99–104; **4.5:** 37–40; **4.6:** 7–8; **4.7:** 93–96; **PE:** 1–10; **AAE:** 19–20, 33. Moved Exercises 4.1: 53–68 to PE.

Chapter 5

- Improved discussion in 5.4 and added new Figure 5.18 to illustrate the Mean Value Theorem.
- Added new Exercises: **5.2:** 33–36; **PE:** 45–46.

Chapter 6

- Clarified cylindrical shell method.
- Converted 6.5 Example 4 to metric units.
- Added introductory discussion of mass distribution along a line, with figure, in 6.6.
- Added new Exercises: **6.1:** 15–16; **6.2:** 45–46; **6.5:** 1–2; **6.6:** 1–6, 19–20; **PE:** 17–18, 35–36.

Chapter 7

- Added explanation for the terminology “indeterminate form.”
- Clarified discussion of separable differential equations in 7.4.
- Replaced \sin^{-1} notation for the inverse sine function with \arcsin as default notation in 7.6, and similarly for other trig functions.
- Added new Exercises: **7.2:** 5–6, 75–76; **7.3:** 5–6, 31–32, 123–128, 149–150; **7.6:** 43–46, 95–96; **AAE:** 9–10, 23.

Chapter 8

- Updated 8.2 Integration by Parts discussion to emphasize $u(x)v'(x) dx$ form rather than $u dv$. Rewrote Examples 1–3 accordingly.
- Removed discussion of tabular integration and associated exercises.
- Updated discussion in 8.5 on how to find constants in the method of partial fractions.
- Updated notation in 8.8 to align with standard usage in statistics.
- Added new Exercises: **8.1:** 41–44; **8.2:** 53–56, 72–73; **8.3:** 75–76; **8.4:** 49–52; **8.5:** 51–66, 73–74; **8.8:** 35–38, 77–78; **PE:** 69–88.

Chapter 9

- Added new Example 3 with Figure 9.3 to illustrate how to construct a slope field.
- Added new Exercises: **9.1:** 11–14; **PE:** 17–22, 43–44.

Chapter 10

- Clarified the differences between a sequence and a series.
- Added new Figure 10.9 to illustrate sum of a series as area of a histogram.
- Added to 10.3 a discussion on the importance of bounding errors in approximations.
- Added new Figure 10.13 illustrating how to use integrals to bound remainder terms of partial sums.
- Rewrote Theorem 10 in 10.4 to bring out similarity to the integral comparison test.
- Added new Figure 10.16 to illustrate the differing behaviors of the harmonic and alternating harmonic series.
- Renamed the n th-Term Test the “ n th-Term Test for Divergence” to emphasize that it says nothing about convergence.
- Added new Figure 10.19 to illustrate polynomials converging to $\ln(1+x)$, which illustrates convergence on the half-open interval $(-1, 1]$.
- Used red dots and intervals to indicate intervals and points where divergence occurs, and blue to indicate convergence, throughout Chapter 10.
- Added new Figure 10.21 to show the six different possibilities for an interval of convergence.
- Added new Exercises: **10.1:** 27–30, 72–77; **10.2:** 19–22, 73–76, 105; **10.3:** 11–12, 39–42; **10.4:** 55–56; **10.5:** 45–46, 65–66; **10.6:** 57–82; **10.7:** 61–65; **10.8:** 23–24, 39–40; **10.9:** 11–12, 37–38; **PE:** 41–44, 97–102.

Chapter 11

- Added new Example 1 and Figure 11.2 in 11.1 to give a straightforward first example of a parametrized curve.
- Updated area formulas for polar coordinates to include conditions for positive r and nonoverlapping θ .
- Added new Example 3 and Figure 11.37 in 11.4 to illustrate intersections of polar curves.
- Added new Exercises: **11.1:** 19–28; **11.2:** 49–50; **11.4:** 21–24.

Chapter 12

- Added new Figure 12.13(b) to show the effect of scaling a vector.
- Added new Example 7 and Figure 12.26 in 12.3 to illustrate projection of a vector.
- Added discussion on general quadric surfaces in 12.6, with new Example 4 and new Figure 12.48 illustrating the description of an ellipsoid not centered at the origin via completing the square.
- Added new Exercises: **12.1:** 31–34, 59–60, 73–76; **12.2:** 43–44; **12.3:** 17–18; **12.4:** 51–57; **12.5:** 49–52.

Chapter 13

- Added sidebars on how to pronounce Greek letters such as kappa, tau, etc.
- Added new Exercises: **13.1:** 1–4, 27–36; **13.2:** 15–16, 19–20; **13.4:** 27–28; **13.6:** 1–2.

Chapter 14

- Elaborated on discussion of open and closed regions in 14.1.
- Standardized notation for evaluating partial derivatives, gradients, and directional derivatives at a point, throughout the chapter.
- Renamed “branch diagrams” as “dependency diagrams,” which clarifies that they capture dependence of variables.
- Added new Exercises: **14.2:** 51–54; **14.3:** 51–54, 59–60, 71–74, 103–104; **14.4:** 20–30, 43–46, 57–58; **14.5:** 41–44; **14.6:** 9–10, 61; **14.7:** 61–62.

Chapter 15

- Added new Figure 15.21b to illustrate setting up limits of a double integral.
- Added new 15.5 Example 1, modified Examples 2 and 3, and added new Figures 15.31, 15.32, and 15.33 to give basic examples of setting up limits of integration for a triple integral.
- Added new material on joint probability distributions as an application of multivariable integration.
- Added new Examples 5, 6 and 7 to Section 15.6.
- Added new Exercises: **15.1:** 15–16, 27–28; **15.6:** 39–44; **15.7:** 1–22.

Chapter 16

- Added new Figure 16.4 to illustrate a line integral of a function.
- Added new Figure 16.17 to illustrate a gradient field.
- Added new Figure 16.18 to illustrate a line integral of a vector field.
- Clarified notation for line integrals in 16.2.
- Added discussion of the sign of potential energy in 16.3.
- Rewrote solution of Example 3 in 16.4 to clarify connection to Green’s Theorem.
- Updated discussion of surface orientation in 16.6 along with Figure 16.52.
- Added new Exercises: **16.2:** 37–38, 41–46; **16.4:** 1–6; **16.6:** 49–50; **16.7:** 1–6; **16.8:** 1–4.

Appendices: Rewrote Appendix A7 on complex numbers.

Continuing Features

Rigor The level of rigor is consistent with that of earlier editions. We continue to distinguish between formal and informal discussions and to point out their differences. Starting with a more intuitive, less formal approach helps students understand a new or difficult concept so they can then appreciate its full mathematical precision and outcomes. We pay attention to defining ideas carefully and to proving theorems appropriate for calculus students, while mentioning deeper or subtler issues they would study in a more advanced course. Our organization and distinctions between informal and formal discussions give the instructor a degree of flexibility in the amount and depth of coverage of the various topics. For example, while we do not prove the Intermediate Value Theorem or the Extreme Value Theorem for continuous functions on a closed finite interval, we do state these theorems precisely, illustrate their meanings in numerous examples, and use them to prove other important results. Furthermore, for those instructors who desire greater depth of coverage, in Appendix 6 we discuss the reliance of these theorems on the completeness of the real numbers.

Writing Exercises Writing exercises placed throughout the text ask students to explore and explain a variety of calculus concepts and applications. In addition, the end of each chapter contains a list of questions for students to review and summarize what they have learned. Many of these exercises make good writing assignments.

End-of-Chapter Reviews and Projects In addition to problems appearing after each section, each chapter culminates with review questions, practice exercises covering the entire chapter, and a series of Additional and Advanced Exercises with more challenging or synthesizing problems. Most chapters also include descriptions of several **Technology Application Projects** that can be worked by individual students or groups of students over a longer period of time. These projects require the use of *Mathematica* or *Maple*, along with pre-made files that are available for download within MyMathLab.

Writing and Applications This text continues to be easy to read, conversational, and mathematically rich. Each new topic is motivated by clear, easy-to-understand examples and is then reinforced by its application to real-world problems of immediate interest to students. A hallmark of this book has been the application of calculus to science and engineering. These applied problems have been updated, improved, and extended continually over the last several editions.

Technology In a course using the text, technology can be incorporated according to the taste of the instructor. Each section contains exercises requiring the use of technology; these are marked with a **T** if suitable for calculator or computer use, or they are labeled **Computer Explorations** if a computer algebra system (CAS, such as *Maple* or *Mathematica*) is required.

Additional Resources

MyMathLab® Online Course (access code required)

Built around Pearson's best-selling content, MyMathLab is an online homework, tutorial, and assessment program designed to work with this text to engage students and improve results. MyMathLab can be successfully implemented in any classroom environment—lab-based, hybrid, fully online, or traditional.

Used by more than 37 million students worldwide, MyMathLab delivers consistent, measurable gains in student learning outcomes, retention, and subsequent course success. Visit www.mymathlab.com/results to learn more.

Preparedness One of the biggest challenges in calculus courses is making sure students are adequately prepared with the prerequisite skills needed to successfully complete their course work. MyMathLab supports students with just-in-time remediation and key-concept review.

- **Integrated Review Course** can be used for just-in-time prerequisite review. These courses contain pre-made quizzes to assess the prerequisite skills needed for each chapter, plus personalized remediation for any gaps in skills that are identified.

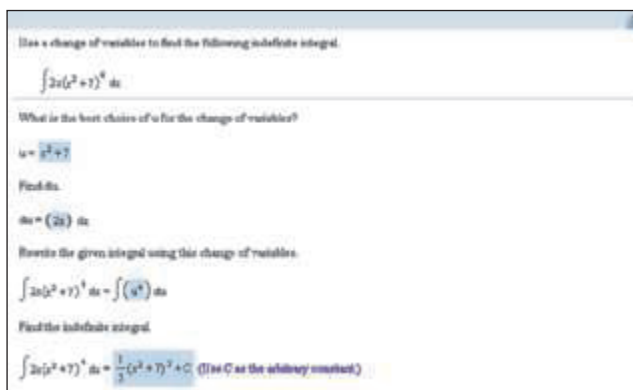


Motivation Students are motivated to succeed when they're engaged in the learning experience and understand the relevance and power of mathematics. MyMathLab's online homework offers students immediate feedback and tutorial assistance that motivates them to do more, which means they retain more knowledge and improve their test scores.

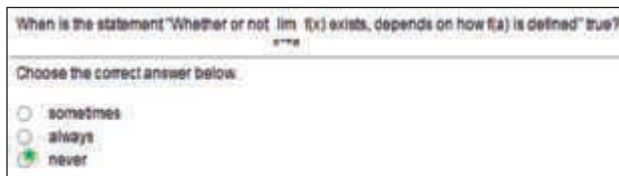
- **Exercises with immediate feedback**—the over 8080 assignable exercises for this text regenerate algorithmically to give students unlimited opportunity for practice and mastery. MyMathLab provides helpful feedback when students enter incorrect answers and includes optional learning aids such as Help Me Solve This, View an Example, videos, and an eText.



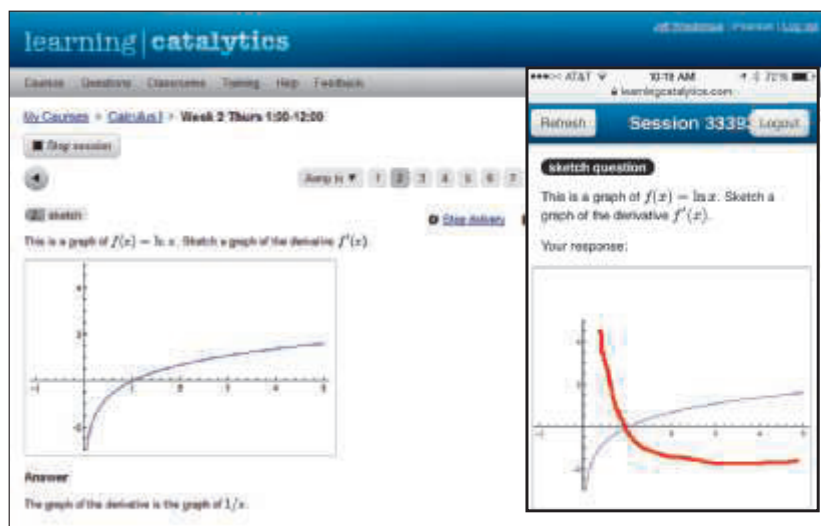
- **Setup and Solve Exercises** ask students to first describe how they will set up and approach the problem. This reinforces students' conceptual understanding of the process they are applying and promotes long-term retention of the skill.



- **Additional Conceptual Questions** focus on deeper, theoretical understanding of the key concepts in calculus. These questions were written by faculty at Cornell University under an NSF grant and are also assignable through Learning Catalytics.

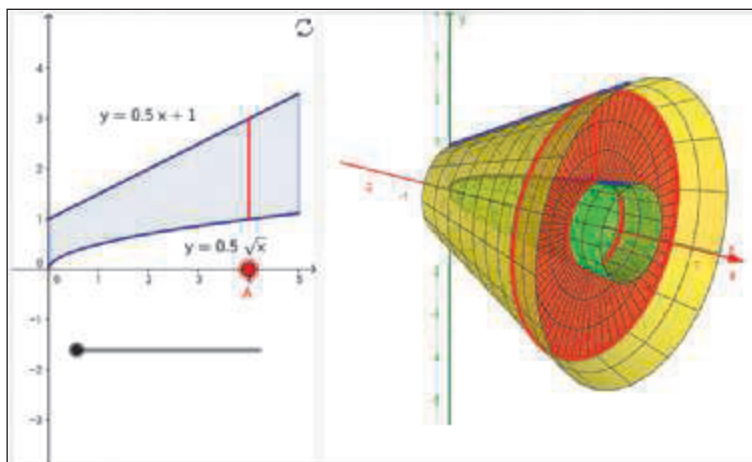


- **Learning Catalytics™** is a student response tool that uses students’ smartphones, tablets, or laptops to engage them in more interactive tasks and thinking during lecture. Learning Catalytics fosters student engagement and peer-to-peer learning with real-time analytics. Learning Catalytics is available to all MyMathLab users.

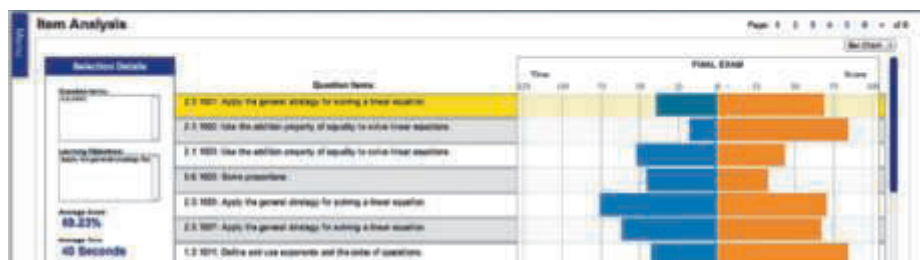


Learning and Teaching Tools

- **Interactive Figures** illustrate key concepts and allow manipulation for use as teaching and learning tools. We also include videos that use the Interactive Figures to explain key concepts.



- **Instructional videos**—hundreds of videos are available as learning aids within exercises and for self-study. The Guide to Video-Based Assignments makes it easy to assign videos for homework by showing which MyMathLab exercises correspond to each video.
- **The complete eText** is available to students through their MyMathLab courses for the lifetime of the edition, giving students unlimited access to the eText within any course using that edition of the text.
- **Enhanced Sample Assignments** These assignments include just-in-time prerequisite review, help keep skills fresh with distributed practice of key concepts, and provide opportunities to work exercises without learning aids so students can check their understanding.
- **PowerPoint Presentations** that cover each section of the book are available for download.
- **Mathematica manual and projects, Maple manual and projects, TI Graphing Calculator manual**—These manuals cover *Maple 17*, *Mathematica 8*, and the TI-84 Plus and TI-89, respectively. Each provides detailed guidance for integrating the software package or graphing calculator throughout the course, including syntax and commands.
- **Accessibility** and achievement go hand in hand. MyMathLab is compatible with the JAWS screen reader, and it enables students to read and interact with multiple-choice and free-response problem types via keyboard controls and math notation input. MyMathLab also works with screen enlargers, including ZoomText, MAGic, and SuperNova. And, all MyMathLab videos have closed-captioning. More information is available at <http://mymathlab.com/accessibility>.
- **A comprehensive gradebook** with enhanced reporting functionality allows you to efficiently manage your course.
 - **The Reporting Dashboard** offers insight as you view, analyze, and report learning outcomes. Student performance data is presented at the class, section, and program levels in an accessible, visual manner so you'll have the information you need to keep your students on track.



- **Item Analysis** tracks class-wide understanding of particular exercises so you can refine your class lectures or adjust the course/department syllabus. Just-in-time teaching has never been easier!

MyMathLab comes from an experienced partner with educational expertise and an eye on the future. Whether you are just getting started with MyMathLab, or have a question along the way, we're here to help you learn about our technologies and how to incorporate them into your course. To learn more about how MyMathLab helps students succeed, visit www.mymathlab.com or contact your Pearson rep.

Instructor's Solutions Manual (downloadable)

ISBN: 0-13-443918-X | 978-0-13-443918-1

The Instructor's Solutions Manual contains complete worked-out solutions to all the exercises in *Thomas' Calculus*. It can be downloaded from within MyMathLab or the Pearson Instructor Resource Center, www.pearsonhighered.com/irc.

Student's Solutions Manual

Single Variable Calculus (Chapters 1–11), ISBN: 0-13-443907-4 | 978-0-13-443907-5

Multivariable Calculus (Chapters 10–16), ISBN: 0-13-443916-3 | 978-0-13-443916-7

The Student's Solutions Manual contains worked-out solutions to all the odd-numbered exercises in *Thomas' Calculus*. These manuals are available in print and can be downloaded from within MyMathLab.

Just-In-Time Algebra and Trigonometry for Calculus, Fourth Edition

ISBN: 0-321-67104-X | 978-0-321-67104-2

Sharp algebra and trigonometry skills are critical to mastering calculus, and *Just-in-Time Algebra and Trigonometry for Calculus* by Guntram Mueller and Ronald I. Brent is designed to bolster these skills while students study calculus. As students make their way through calculus, this brief supplementary text is with them every step of the way, showing them the necessary algebra or trigonometry topics and pointing out potential problem spots. The easy-to-use table of contents has topics arranged in the order in which students will need them as they study calculus. This supplement is available in printed form only (note that MyMathLab contains a separate diagnostic and remediation system for gaps in algebra and trigonometry skills).

Technology Manuals and Projects (downloadable)*Maple Manual and Projects* by Marie Vanisko, Carroll College*Mathematica Manual and Projects* by Marie Vanisko, Carroll College*TI-Graphing Calculator Manual* by Elaine McDonald-Newman, Sonoma State University

These manuals and projects cover Maple 17, Mathematica 9, and the TI-84 Plus and TI-89. Each manual provides detailed guidance for integrating a specific software package or graphing calculator throughout the course, including syntax and commands. The projects include instructions and ready-made application files for Maple and Mathematica. These materials are available to download within MyMathLab.

TestGen®

ISBN: 0-13-443922-8 | 978-0-13-443922-8

TestGen® (www.pearsoned.com/testgen) enables instructors to build, edit, print, and administer tests using a computerized bank of questions developed to cover all the objectives of the text. TestGen is algorithmically based, allowing instructors to create multiple but equivalent versions of the same question or test with the click of a button. Instructors can also modify test bank questions or add new questions. The software and test bank are available for download from Pearson Education's online catalog, www.pearsonhighered.com.

PowerPoint® Lecture Slides

ISBN: 0-13-443911-2 | 978-0-13-443911-2

These classroom presentation slides were created for the *Thomas' Calculus* series. Key graphics from the book are included to help bring the concepts alive in the classroom. These files are available to qualified instructors through the Pearson Instructor Resource Center, www.pearsonhighered.com/irc, and within MyMathLab.

Acknowledgments

We are grateful to Duane Kouba, who created many of the new exercises. We would also like to express our thanks to the people who made many valuable contributions to this edition as it developed through its various stages:

Accuracy Checkers

Thomas Wegleitner
Jennifer Blue
Lisa Collette

Reviewers for the Fourteenth Edition

Alessandro Arsie, *University of Toledo*
Doug Baldwin, *SUNY Geneseo*
Steven Heilman, *UCLA*
David Horntrop, *New Jersey Institute of Technology*
Eric B. Kahn, *Bloomsburg University*
Colleen Kirk, *California Polytechnic State University*
Mark McConnell, *Princeton University*

Niels Martin Møller, *Princeton University*
James G. O'Brien, *Wentworth Institute of Technology*
Alan Saleski, *Loyola University Chicago*
Alan Von Hermann, *Santa Clara University*
Don Gayan Wilathgamuwa, *Montana State University*
James Wilson, *Iowa State University*

The following faculty members provided direction on the development of the MyMathLab course for this edition.

Charles Obare, *Texas State Technical College, Harlingen*
Elmira Yakutova-Lorentz, *Eastern Florida State College*
C. Sohn, *SUNY Geneseo*
Ksenia Owens, *Napa Valley College*
Ruth Mortha, *Malcolm X College*
George Reuter, *SUNY Geneseo*
Daniel E. Osborne, *Florida A&M University*
Luis Rodriguez, *Miami Dade College*
Abbas Meigooni, *Lincoln Land Community College*
Nader Yassin, *Del Mar College*
Arthur J. Rosenthal, *Salem State University*
Valerie Bouagnon, *DePaul University*
Brooke P. Quinlan, *Hillsborough Community College*
Shuvra Gupta, *Iowa State University*
Alexander Casti, *Farleigh Dickinson University*
Sharda K. Gudehithlu, *Wilbur Wright College*
Deanna Robinson, *McLennan Community College*

Kai Chuang, *Central Arizona College*
Vandana Srivastava, *Pitt Community College*
Brian Albright, *Concordia University*
Brian Hayes, *Triton College*
Gabriel Cuarenta, *Merced College*
John Beyers, *University of Maryland University College*
Daniel Pellegrini, *Triton College*
Debra Johnsen, *Orangeburg Calhoun Technical College*
Olga Tsukernik, *Rochester Institute of Technology*
Jorge Sarmiento, *County College of Morris*
Val Mohanakumar, *Hillsborough Community College*
MK Panahi, *El Centro College*
Sabrina Ripp, *Tulsa Community College*
Mona Panchal, *East Los Angeles College*
Gail Illich, *McLennan Community College*
Mark Farag, *Farleigh Dickinson University*
Selena Mohan, *Cumberland County College*

Dedication

We regret that prior to the writing of this edition our coauthor Maurice Weir passed away. Maury was dedicated to achieving the highest possible standards in the presentation of mathematics. He insisted on clarity, rigor, and readability. Maury was a role model to his students, his colleagues, and his coauthors. He was very proud of his daughters, Maia Coyle and Renee Waina, and of his grandsons, Matthew Ryan and Andrew Dean Waina. He will be greatly missed.

1

Functions



OVERVIEW Functions are fundamental to the study of calculus. In this chapter we review what functions are and how they are visualized as graphs, how they are combined and transformed, and ways they can be classified.

1.1 Functions and Their Graphs

Functions are a tool for describing the real world in mathematical terms. A function can be represented by an equation, a graph, a numerical table, or a verbal description; we will use all four representations throughout this book. This section reviews these ideas.

Functions; Domain and Range

The temperature at which water boils depends on the elevation above sea level. The interest paid on a cash investment depends on the length of time the investment is held. The area of a circle depends on the radius of the circle. The distance an object travels depends on the elapsed time.

In each case, the value of one variable quantity, say y , depends on the value of another variable quantity, which we often call x . We say that “ y is a function of x ” and write this symbolically as

$$y = f(x) \quad (\text{“}y \text{ equals } f \text{ of } x\text{”}).$$

The symbol f represents the function, the letter x is the **independent variable** representing the input value to f , and y is the **dependent variable** or output value of f at x .

DEFINITION A **function** f from a set D to a set Y is a rule that assigns a *unique* value $f(x)$ in Y to each x in D .

The set D of all possible input values is called the **domain** of the function. The set of all output values of $f(x)$ as x varies throughout D is called the **range** of the function. The range might not include every element in the set Y . The domain and range of a function can be any sets of objects, but often in calculus they are sets of real numbers interpreted as points of a coordinate line. (In Chapters 13–16, we will encounter functions for which the elements of the sets are points in the plane, or in space.)

Often a function is given by a formula that describes how to calculate the output value from the input variable. For instance, the equation $A = \pi r^2$ is a rule that calculates the area A of a circle from its radius r . When we define a function $y = f(x)$ with a formula and the domain is not stated explicitly or restricted by context, the domain is assumed to

be the largest set of real x -values for which the formula gives real y -values. This is called the **natural domain** of f . If we want to restrict the domain in some way, we must say so. The domain of $y = x^2$ is the entire set of real numbers. To restrict the domain of the function to, say, positive values of x , we would write “ $y = x^2, x > 0$.”

Changing the domain to which we apply a formula usually changes the range as well. The range of $y = x^2$ is $[0, \infty)$. The range of $y = x^2, x \geq 2$, is the set of all numbers obtained by squaring numbers greater than or equal to 2. In set notation (see Appendix 1), the range is $\{x^2 | x \geq 2\}$ or $\{y | y \geq 4\}$ or $[4, \infty)$.

When the range of a function is a set of real numbers, the function is said to be **real-valued**. The domains and ranges of most real-valued functions we consider are intervals or combinations of intervals. Sometimes the range of a function is not easy to find.

A function f is like a machine that produces an output value $f(x)$ in its range whenever we feed it an input value x from its domain (Figure 1.1). The function keys on a calculator give an example of a function as a machine. For instance, the \sqrt{x} key on a calculator gives an output value (the square root) whenever you enter a nonnegative number x and press the \sqrt{x} key.

A function can also be pictured as an **arrow diagram** (Figure 1.2). Each arrow associates to an element of the domain D a single element in the set Y . In Figure 1.2, the arrows indicate that $f(a)$ is associated with a , $f(x)$ is associated with x , and so on. Notice that a function can have the same *output value* for two different input elements in the domain (as occurs with $f(a)$ in Figure 1.2), but each input element x is assigned a *single* output value $f(x)$.



FIGURE 1.1 A diagram showing a function as a kind of machine.

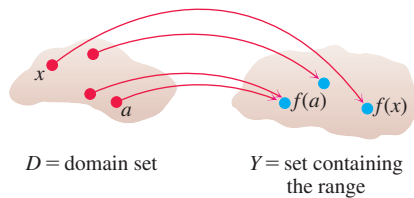


FIGURE 1.2 A function from a set D to a set Y assigns a unique element of Y to each element in D .

EXAMPLE 1 Verify the natural domains and associated ranges of some simple functions. The domains in each case are the values of x for which the formula makes sense.

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Solution The formula $y = x^2$ gives a real y -value for any real number x , so the domain is $(-\infty, \infty)$. The range of $y = x^2$ is $[0, \infty)$ because the square of any real number is nonnegative and every nonnegative number y is the square of its own square root: $y = (\sqrt{y})^2$ for $y \geq 0$.

The formula $y = 1/x$ gives a real y -value for every x except $x = 0$. For consistency in the rules of arithmetic, *we cannot divide any number by zero*. The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers, is the set of all nonzero real numbers, since $y = 1/(1/y)$. That is, for $y \neq 0$ the number $x = 1/y$ is the input that is assigned to the output value y .

The formula $y = \sqrt{x}$ gives a real y -value only if $x \geq 0$. The range of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number's square root (namely, it is the square root of its own square).

In $y = \sqrt{4 - x}$, the quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$, or $x \leq 4$. The formula gives nonnegative real y -values for all $x \leq 4$. The range of $\sqrt{4 - x}$ is $[0, \infty)$, the set of all nonnegative numbers.

The formula $y = \sqrt{1 - x^2}$ gives a real y -value for every x in the closed interval from -1 to 1 . Outside this domain, $1 - x^2$ is negative and its square root is not a real number. The values of $1 - x^2$ vary from 0 to 1 on the given domain, and the square roots of these values do the same. The range of $\sqrt{1 - x^2}$ is $[0, 1]$. ■

Graphs of Functions

If f is a function with domain D , its **graph** consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f . In set notation, the graph is

$$\{(x, f(x)) \mid x \in D\}.$$

The graph of the function $f(x) = x + 2$ is the set of points with coordinates (x, y) for which $y = x + 2$. Its graph is the straight line sketched in Figure 1.3.

The graph of a function f is a useful picture of its behavior. If (x, y) is a point on the graph, then $y = f(x)$ is the height of the graph above (or below) the point x . The height may be positive or negative, depending on the sign of $f(x)$ (Figure 1.4).

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

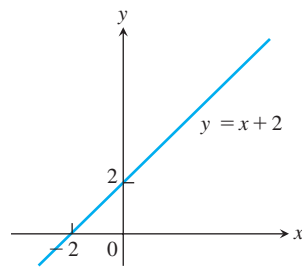


FIGURE 1.3 The graph of $f(x) = x + 2$ is the set of points (x, y) for which y has the value $x + 2$.

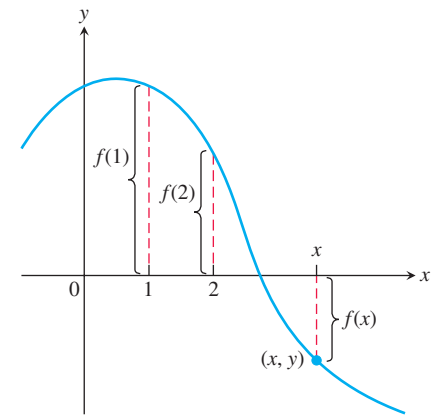


FIGURE 1.4 If (x, y) lies on the graph of f , then the value $y = f(x)$ is the height of the graph above the point x (or below x if $f(x)$ is negative).

EXAMPLE 2 Graph the function $y = x^2$ over the interval $[-2, 2]$.

Solution Make a table of xy -pairs that satisfy the equation $y = x^2$. Plot the points (x, y) whose coordinates appear in the table, and draw a *smooth* curve (labeled with its equation) through the plotted points (see Figure 1.5). ■

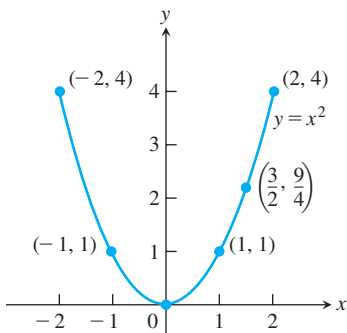
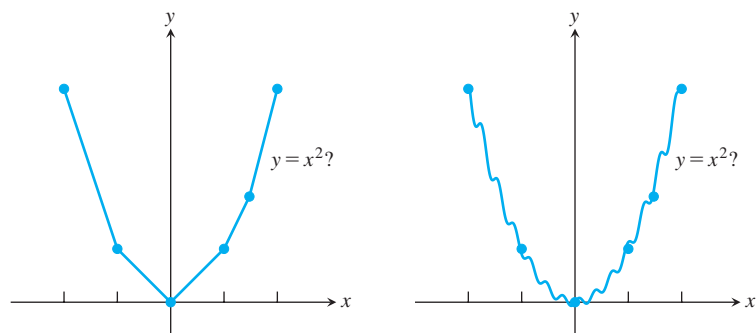


FIGURE 1.5 Graph of the function in Example 2.

How do we know that the graph of $y = x^2$ doesn't look like one of these curves?



To find out, we could plot more points. But how would we then connect *them*? The basic question still remains: How do we know for sure what the graph looks like between the points we plot? Calculus answers this question, as we will see in Chapter 4. Meanwhile, we will have to settle for plotting points and connecting them as best we can.

Representing a Function Numerically

We have seen how a function may be represented algebraically by a formula and visually by a graph (Example 2). Another way to represent a function is **numerically**, through a table of values. Numerical representations are often used by engineers and experimental scientists. From an appropriate table of values, a graph of the function can be obtained using the method illustrated in Example 2, possibly with the aid of a computer. The graph consisting of only the points in the table is called a **scatterplot**.

EXAMPLE 3 Musical notes are pressure waves in the air. The data associated with Figure 1.6 give recorded pressure displacement versus time in seconds of a musical note produced by a tuning fork. The table provides a representation of the pressure function (in micropascals) over time. If we first make a scatterplot and then connect the data points (t, p) from the table, we obtain the graph shown in the figure.

Time	Pressure	Time	Pressure
0.00091	-0.080	0.00362	0.217
0.00108	0.200	0.00379	0.480
0.00125	0.480	0.00398	0.681
0.00144	0.693	0.00416	0.810
0.00162	0.816	0.00435	0.827
0.00180	0.844	0.00453	0.749
0.00198	0.771	0.00471	0.581
0.00216	0.603	0.00489	0.346
0.00234	0.368	0.00507	0.077
0.00253	0.099	0.00525	-0.164
0.00271	-0.141	0.00543	-0.320
0.00289	-0.309	0.00562	-0.354
0.00307	-0.348	0.00579	-0.248
0.00325	-0.248	0.00598	-0.035
0.00344	-0.041		

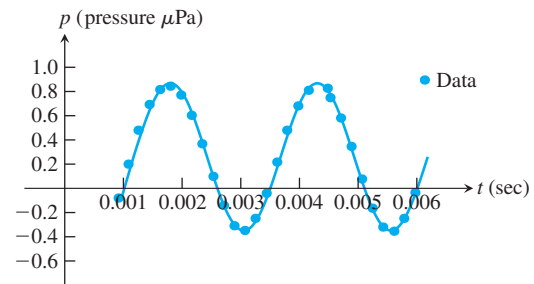


FIGURE 1.6 A smooth curve through the plotted points gives a graph of the pressure function represented by the accompanying tabled data (Example 3).

The Vertical Line Test for a Function

Not every curve in the coordinate plane can be the graph of a function. A function f can have only one value $f(x)$ for each x in its domain, so *no vertical* line can intersect the graph of a function more than once. If a is in the domain of the function f , then the vertical line $x = a$ will intersect the graph of f at the single point $(a, f(a))$.

A circle cannot be the graph of a function, since some vertical lines intersect the circle twice. The circle graphed in Figure 1.7a, however, contains the graphs of two functions of x , namely the upper semicircle defined by the function $f(x) = \sqrt{1 - x^2}$ and the lower semicircle defined by the function $g(x) = -\sqrt{1 - x^2}$ (Figures 1.7b and 1.7c).

Piecewise-Defined Functions

Sometimes a function is described in pieces by using different formulas on different parts of its domain. One example is the **absolute value function**

$$|x| = \begin{cases} x, & x \geq 0 & \text{First formula} \\ -x, & x < 0 & \text{Second formula} \end{cases}$$

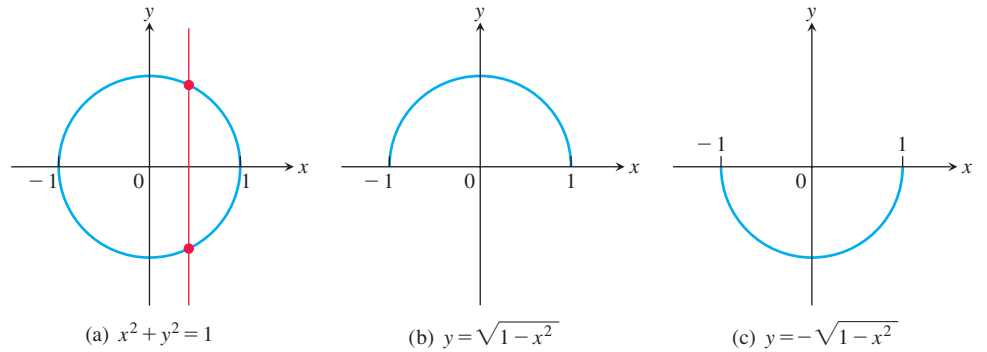


FIGURE 1.7 (a) The circle is not the graph of a function; it fails the vertical line test. (b) The upper semicircle is the graph of the function $f(x) = \sqrt{1 - x^2}$. (c) The lower semicircle is the graph of the function $g(x) = -\sqrt{1 - x^2}$.

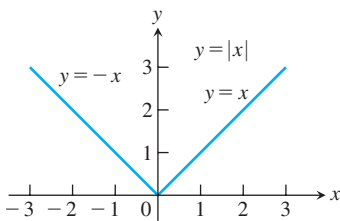


FIGURE 1.8 The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.

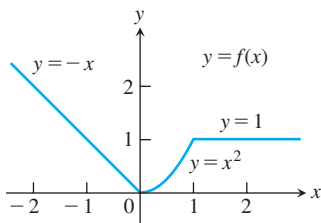


FIGURE 1.9 To graph the function $y = f(x)$ shown here, we apply different formulas to different parts of its domain (Example 4).

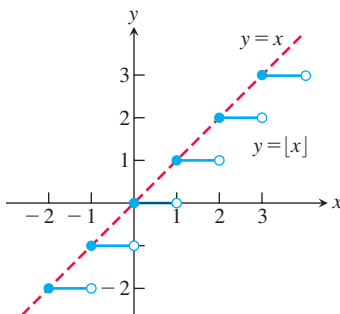


FIGURE 1.10 The graph of the greatest integer function $y = \lfloor x \rfloor$ lies on or below the line $y = x$, so it provides an integer floor for x (Example 5).

whose graph is given in Figure 1.8. The right-hand side of the equation means that the function equals x if $x \geq 0$, and equals $-x$ if $x < 0$. Piecewise-defined functions often arise when real-world data are modeled. Here are some other examples.

EXAMPLE 4 The function

$$f(x) = \begin{cases} -x, & x < 0 & \text{First formula} \\ x^2, & 0 \leq x \leq 1 & \text{Second formula} \\ 1, & x > 1 & \text{Third formula} \end{cases}$$

is defined on the entire real line but has values given by different formulas, depending on the position of x . The values of f are given by $y = -x$ when $x < 0$, $y = x^2$ when $0 \leq x \leq 1$, and $y = 1$ when $x > 1$. The function, however, is *just one function* whose domain is the entire set of real numbers (Figure 1.9).

EXAMPLE 5 The function whose value at any number x is the *greatest integer less than or equal to* x is called the **greatest integer function** or the **integer floor function**. It is denoted $\lfloor x \rfloor$. Figure 1.10 shows the graph. Observe that

$$\begin{aligned} \lfloor 2.4 \rfloor &= 2, & \lfloor 1.9 \rfloor &= 1, & \lfloor 0 \rfloor &= 0, & \lfloor -1.2 \rfloor &= -2, \\ \lfloor 2 \rfloor &= 2, & \lfloor 0.2 \rfloor &= 0, & \lfloor -0.3 \rfloor &= -1, & \lfloor -2 \rfloor &= -2. \end{aligned}$$

EXAMPLE 6 The function whose value at any number x is the *smallest integer greater than or equal to* x is called the **least integer function** or the **integer ceiling function**. It is denoted $\lceil x \rceil$. Figure 1.11 shows the graph. For positive values of x , this function might represent, for example, the cost of parking x hours in a parking lot that charges \$1 for each hour or part of an hour.

Increasing and Decreasing Functions

If the graph of a function climbs or rises as you move from left to right, we say that the function is *increasing*. If the graph descends or falls as you move from left to right, the function is *decreasing*.

DEFINITIONS Let f be a function defined on an interval I and let x_1 and x_2 be two distinct points in I .

1. If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I .
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I .

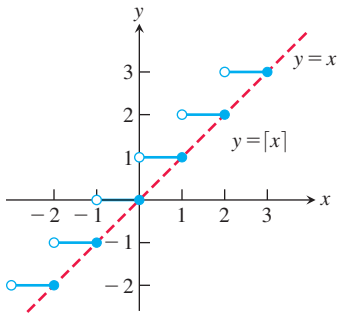


FIGURE 1.11 The graph of the least integer function $y = [x]$ lies on or above the line $y = x$, so it provides an integer ceiling for x (Example 6).

It is important to realize that the definitions of increasing and decreasing functions must be satisfied for *every* pair of points x_1 and x_2 in I with $x_1 < x_2$. Because we use the inequality $<$ to compare the function values, instead of \leq , it is sometimes said that f is *strictly* increasing or decreasing on I . The interval I may be finite (also called bounded) or infinite (unbounded).

EXAMPLE 7 The function graphed in Figure 1.9 is decreasing on $(-\infty, 0)$ and increasing on $(0, 1)$. The function is neither increasing nor decreasing on the interval $(1, \infty)$ because the function is constant on that interval, and hence the strict inequalities in the definition of increasing or decreasing are not satisfied on $(1, \infty)$. ■

Even Functions and Odd Functions: Symmetry

The graphs of *even* and *odd* functions have special symmetry properties.

DEFINITIONS A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,

odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.

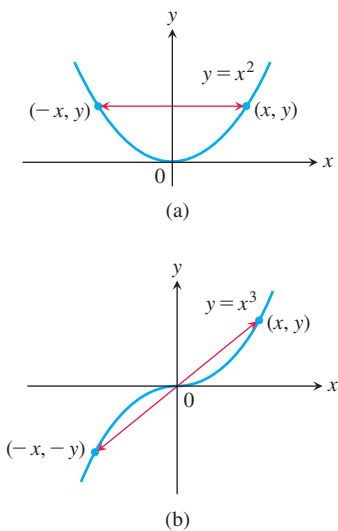


FIGURE 1.12 (a) The graph of $y = x^2$ (an even function) is symmetric about the y -axis. (b) The graph of $y = x^3$ (an odd function) is symmetric about the origin.

The names *even* and *odd* come from powers of x . If y is an even power of x , as in $y = x^2$ or $y = x^4$, it is an even function of x because $(-x)^2 = x^2$ and $(-x)^4 = x^4$. If y is an odd power of x , as in $y = x$ or $y = x^3$, it is an odd function of x because $(-x)^1 = -x$ and $(-x)^3 = -x^3$.

The graph of an even function is **symmetric about the y -axis**. Since $f(-x) = f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, y)$ lies on the graph (Figure 1.12a). A reflection across the y -axis leaves the graph unchanged.

The graph of an odd function is **symmetric about the origin**. Since $f(-x) = -f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, -y)$ lies on the graph (Figure 1.12b). Equivalently, a graph is symmetric about the origin if a rotation of 180° about the origin leaves the graph unchanged. Notice that the definitions imply that both x and $-x$ must be in the domain of f .

EXAMPLE 8 Here are several functions illustrating the definitions.

$f(x) = x^2$

Even function: $(-x)^2 = x^2$ for all x ; symmetry about y -axis. So $f(-3) = 9 = f(3)$. Changing the sign of x does not change the value of an even function.

$f(x) = x^2 + 1$

Even function: $(-x)^2 + 1 = x^2 + 1$ for all x ; symmetry about y -axis (Figure 1.13a).

$f(x) = x$

Odd function: $(-x) = -x$ for all x ; symmetry about the origin. So $f(-3) = -3$ while $f(3) = 3$. Changing the sign of x changes the sign of an odd function.

$f(x) = x + 1$

Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. The two are not equal.

Not even: $(-x) + 1 \neq x + 1$ for all $x \neq 0$ (Figure 1.13b). ■

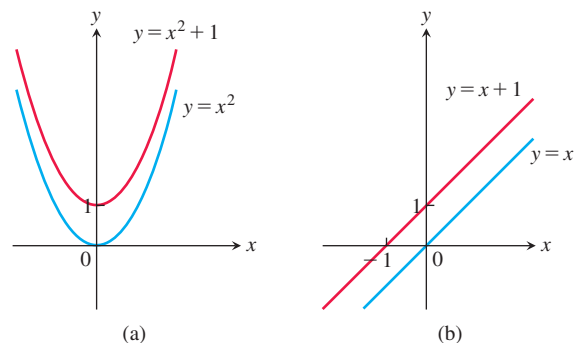


FIGURE 1.13 (a) When we add the constant term 1 to the function $y = x^2$, the resulting function $y = x^2 + 1$ is still even and its graph is still symmetric about the y -axis. (b) When we add the constant term 1 to the function $y = x$, the resulting function $y = x + 1$ is no longer odd, since the symmetry about the origin is lost. The function $y = x + 1$ is also not even (Example 8).

Common Functions

A variety of important types of functions are frequently encountered in calculus.

Linear Functions A function of the form $f(x) = mx + b$, where m and b are fixed constants, is called a **linear function**. Figure 1.14a shows an array of lines $f(x) = mx$. Each of these has $b = 0$, so these lines pass through the origin. The function $f(x) = x$ where $m = 1$ and $b = 0$ is called the **identity function**. Constant functions result when the slope is $m = 0$ (Figure 1.14b).

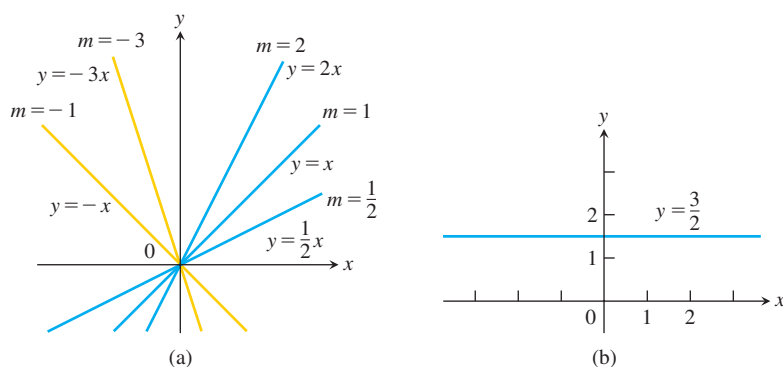


FIGURE 1.14 (a) Lines through the origin with slope m . (b) A constant function with slope $m = 0$.

DEFINITION Two variables y and x are **proportional** (to one another) if one is always a constant multiple of the other—that is, if $y = kx$ for some nonzero constant k .

If the variable y is proportional to the reciprocal $1/x$, then sometimes it is said that y is **inversely proportional** to x (because $1/x$ is the multiplicative inverse of x).

Power Functions A function $f(x) = x^a$, where a is a constant, is called a **power function**. There are several important cases to consider.

(a) $f(x) = x^a$ with $a = n$, a positive integer.

The graphs of $f(x) = x^n$, for $n = 1, 2, 3, 4, 5$, are displayed in Figure 1.15. These functions are defined for all real values of x . Notice that as the power n gets larger, the curves tend to flatten toward the x -axis on the interval $(-1, 1)$ and to rise more steeply for $|x| > 1$. Each curve passes through the point $(1, 1)$ and through the origin. The graphs of functions with even powers are symmetric about the y -axis; those with odd powers are symmetric about the origin. The even-powered functions are decreasing on the interval $(-\infty, 0]$ and increasing on $[0, \infty)$; the odd-powered functions are increasing over the entire real line $(-\infty, \infty)$.

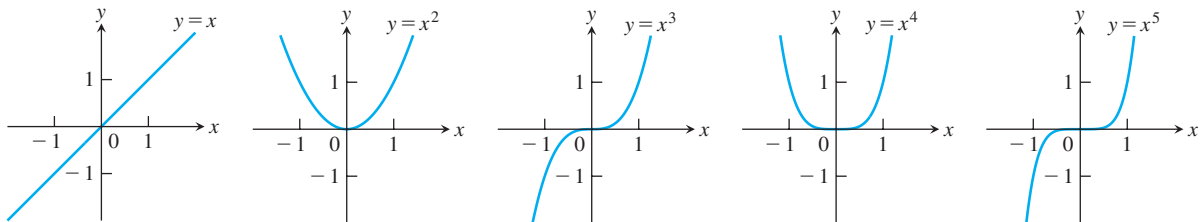


FIGURE 1.15 Graphs of $f(x) = x^n$, $n = 1, 2, 3, 4, 5$, defined for $-\infty < x < \infty$.

(b) $f(x) = x^a$ with $a = -1$ or $a = -2$.

The graphs of the functions $f(x) = x^{-1} = 1/x$ and $g(x) = x^{-2} = 1/x^2$ are shown in Figure 1.16. Both functions are defined for all $x \neq 0$ (you can never divide by zero). The graph of $y = 1/x$ is the hyperbola $xy = 1$, which approaches the coordinate axes far from the origin. The graph of $y = 1/x^2$ also approaches the coordinate axes. The graph of the function f is symmetric about the origin; f is decreasing on the intervals $(-\infty, 0)$ and $(0, \infty)$. The graph of the function g is symmetric about the y -axis; g is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

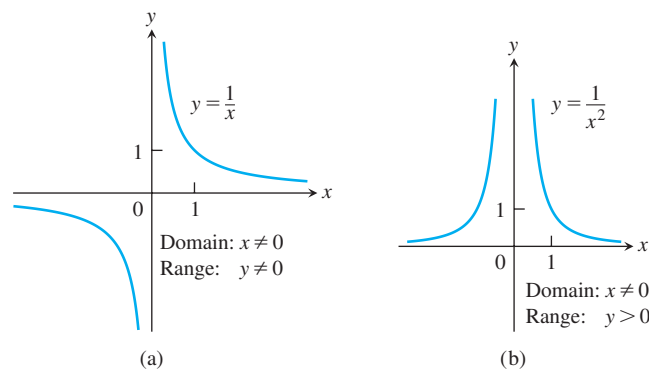


FIGURE 1.16 Graphs of the power functions $f(x) = x^a$. (a) $a = -1$, (b) $a = -2$.

(c) $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2},$ and $\frac{2}{3}$.

The functions $f(x) = x^{1/2} = \sqrt{x}$ and $g(x) = x^{1/3} = \sqrt[3]{x}$ are the **square root** and **cube root** functions, respectively. The domain of the square root function is $[0, \infty)$, but the cube root function is defined for all real x . Their graphs are displayed in Figure 1.17, along with the graphs of $y = x^{3/2}$ and $y = x^{2/3}$. (Recall that $x^{3/2} = (x^{1/2})^3$ and $x^{2/3} = (x^{1/3})^2$.)

Polynomials A function p is a **polynomial** if

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \dots, a_n$ are real constants (called the **coefficients** of the polynomial). All polynomials have domain $(-\infty, \infty)$. If the